International Journal of Theoretical Physics, Vol. 34, No. 7, 1995

Noncommutative Version of Nikodym Boundedness Theorem for Uniform Space-Valued Functions

Paolo de Lucia[1] and Endre Pap[2]

Received February 23, 1995

A Nikodym boundedness-type theorem with necessary and sufficient conditions for a family of functions defined on a $\sigma(\oplus)$–difference-poset and with values in a uniform space is proved. For a special important case—orthomodular lattice—the conditions are relaxed.

1. INTRODUCTION

The Nikodym boundedness theorem is one of the most important theorems in commutative measure theory. It says that a family $\mathcal{M}$ of countable additive signed measures m, defined on a σ-algebra Σ, which is pointwise bounded, i.e., for each $E \in \Sigma$ there exists $M_E > 0$ such that $|m(E)| < M_E$ $(m \in \mathcal{M})$, is uniformly bounded, i.e., there exists $M > 0$ such that

$$|m(E)| < M \qquad (m \in \mathcal{M}, E \in \Sigma)$$

(Dunford and Swartz, 1958).

The noncommutative measure-theoretic version of this theorem consists in the generalization of the domain of the considered measures to a quantum logic, so that the measures are now states (Dvurečenskij, 1993; Cook, 1978). Usually in the quantum logic approach to quantum mechanics the states are represented by probability measures on orthomodular lattices (Kalmbach, 1983; Ptak and Pulmannová, 1991; Dvurečenskij, 1993). But in this situation, in contrast to the Boolean case, it can happen that there does not exist any nontrivial probability, even a group-valued measure (Navara, n.d.).

[1]Dipartimento Matematica e Applicazioni, Universita Federico II, 80126 Naples, Italy.
[2]Institute of Mathematics, University of Novi Sad, 21 000 Novi Sad, Yugoslavia.

0020-7748/95/0700-0981$07.50/0

Very recently, motivated by investigations in the theory of fuzzy sets, Chovanec and Kopka have introduced the new, very useful notion of difference-poset (D-poset) (Kôpka, 1992; Kôpka and Chovanec, 1994). This structure is a generalization of quantum logic (Kalmbach, 1983; Ptak and Pulmannova, 1991), MV-algebras (Mundici, 1986), orthoalgebras (Foulis *et al.*, 1992), the set of all effects (Dvurečenskij, n.d.), and the collection of fuzzy sets (Kôpka, 1992).

The Nikodym boundedness theorem was generalized not only with respect to the domain of the measure, but also with respect to the range and the properties of the considered measures. So there are vector-, group-, and even semigroup-valued measure versions of the Nikodym boundedness theorem. On the other hand, interest in nonadditive set functions (submeasures, k-triangular set functions, decomposable measures, null-additive set functions) has grown in view of many different applications. There are recent investigations of set functions with values in sets endowed only with some topological structures without any algebraic operations: metric space (Pap, 1988), special topological spaces (Klimkin, 1989), and uniform spaces (Pap, 1991a,b).

The classical proof of the Nikodym boundedness theorem (see, for example, Dunford and Swartz, 1958, pp. 156–160) works with the Bair category theorem using the transformation of the measure space into a complete metric space. A disadvantage of this approach is the need for the completeness of the considered metric space and also the distributivity in the domain, which are not usually satisfied for noncommutative versions of Nikodym boundedness theorems. A more convenient approach in this case is the so-called "sliding hump" technique, used by Lebesgue and Toeplitz. This method was recently unified and generalized in many papers with many applications in functional analysis and measure theory; see Pap (1982), Antosik and Swartz (1985), and Swartz (1992) and the survey article of de Lucia and Pap (n.d.). Applications of this method can be found in Cook (1978), Guarigilia (1990), and Pap (1986).

In this paper we continue our investigations, which were started in de Lucia and Pap (1995), of functions defined on difference-posets and with the values in an arbitrary uniform space.

2. DIFFERENCE-POSETS

In this section we explain some notions and their properties which we need in the following section.

We have by Kôpka (1992), Kôpka and Chovanec (1994), and Dvurečenskij and Riečan (1994), the definition of the basic structure.

Definition 1. A D-poset (difference poset) is a partially ordered set $\mathbf{L}$ with a partial ordering $\leq$, maximal element $\mathbf{1}$, and a partial binary operation $\ominus$: $\mathbf{L} \times \mathbf{L} \to \mathbf{L}$, called difference, such that, for $a, b \in \mathbf{L}$, $b \ominus a$ is defined if and only if $a \leq b$, for which the following axioms hold for $a, b, c \in \mathbf{L}$:

(DP_1) $b \ominus a \leq b$.
(DP_2) $b \ominus (b \ominus a) = a$.
(DP_3) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

These axioms imply that there exists also a minimal element $\mathbf{0}$ $(= \mathbf{1} \ominus \mathbf{1})$.

The following properties of the operation $\ominus$ have been proved in Kopka and Chovanec (1994):

(a) $a \ominus \mathbf{0} = a$.
(b) $a \ominus a = \mathbf{0}$.
(c) $a \leq b \Rightarrow b \ominus a = \mathbf{0} \Leftrightarrow b = a$.
(d) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = \mathbf{0}$.
(e) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(f) $b \leq c, a \leq c \ominus b \Rightarrow b \leq c \ominus a$, and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(g) $a \leq b \leq c \Rightarrow a \leq (c \ominus (b \ominus a)) \ominus a = c \ominus b$.

For an arbitrary but fixed element $a \in \mathbf{L}$ we define

$$a^{\perp} := \mathbf{1} \ominus a$$

We have:

(i) $a^{\perp\perp} = a$.
(ii) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.

The elements a and b from $\mathbf{L}$ are orthogonal, denoted by $a \perp b$, iff $a \leq b^{\perp}$ (or $b \leq a^{\perp}$).

We define a partial binary operation $\oplus$: $\mathbf{L} \times \mathbf{L} \to \mathbf{L}$ for orthogonal elements a and b such that

$$b \leq a \oplus b \qquad \text{and} \qquad a = (a \oplus b) \ominus b$$

This operation $\oplus$ is commutative and associative (Dvurečenskij and Riečan, 1994).

The notion of D-poset covers many important examples.

Example 1 (de Lucia and Dvurečenskij, 1993a; de Lucia and Morales, 1992; de Lucia and Pap, 1995; Dvurečenskij and Pulmannova, 1994a,b; Ptak and Pulmannova, 1991). An orthomodular poset is a partially ordered set $\mathbf{O}$

with an ordering $\leq$, the minimal and maximal elements **0** and **1**, respectively, and an orthocomplementation $'$: $\mathbf{O} \to \mathbf{O}$ such that:

(OM_1) $a'' = a$ ($a \in \mathbf{O}$).
(OM_2) $a \vee a' = \mathbf{1}$ ($a \in \mathbf{O}$).
(OM_3) If $a \leq b$, then $b' \leq a'$.
(OM_4) If $a \leq b'$, then $a \vee b \in \mathbf{O}$.
(OM_5) If $a \leq b$, then $b = a \vee (a \vee b')'$.

Taking for $a \leq b$

$$b \ominus a := (a \vee b')'$$

we obtain that the orthomodular poset **O** is a D-poset.

Example 2 (Chang, 1958; Mundici, 1986). An MV-algebra is a set **M** endowed with two binary operations $\oplus$ and $\odot$, a unary operation $*$, and two elements 0 and 1 such that, for all $a, b, c \in \mathbf{M}$:

(MV_1) $a \oplus b = b \oplus a$.
(MV_2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(MV_3) $a \oplus 0 = a$.
(MV_4) $a \oplus 1 = 1$.
(MV_5) $(a^*)^* = a$.
(MV_6) $0^* = 1$.
(MV_7) $a \oplus a^* = 1$.
(MV_8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.
(MV_9) $a \odot b = (a^* \oplus b^*)^*$.

Taking

$$a \leq b \Leftrightarrow (a \odot b^*) \oplus b = b$$

and for $a \leq b$

$$b \ominus a := (a \oplus b^*)^*$$

we obtain that the MV-algebra **M** is a D-poset.

Example 3 (Foulis *et al.*, 1992; Dvurečenskij and Riečan, 1994). An orthoalgebra is a set **A** with two particular elements **0**, **1** and with a partial binary operation $\oplus$: $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ such that for all $a, b, c \in \mathbf{A}$:

(OA_1) If $a \oplus b \in \mathbf{A}$, then $b \oplus a \in \mathbf{A}$ and $a \oplus b = b \oplus a$.
(OA_2) If $b \oplus c \in \mathbf{A}$ and $a \oplus (b \oplus c) \in \mathbf{A}$, then $a \oplus b \in A$ and $(a \oplus b) \oplus c \in \mathbf{A}$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(OA_3) For any $a \in \mathbf{A}$ there is a unique $b \in \mathbf{A}$ such that $a \oplus b$ is defined, and $a \oplus b = \mathbf{1}$.

(OA_4) If $a \oplus a$ is defined, then $a = \mathbf{0}$.

We have $a \leq b$ iff there exists an element $c \in \mathbf{A}$ such that $a \oplus c$ is defined in $\mathbf{A}$ and $a \oplus c = b$. An element b is the orthocomplement of a (denoted by $a^{\perp}$) iff b is a (unique) element of $\mathbf{A}$ such that $b \oplus a$ is defined in $\mathbf{A}$ and $a \oplus b = \mathbf{1}$.

Taking for $a \leq b$

$$b \ominus a := (a \oplus b^{\perp})^{\perp}$$

we obtain that the orthoalgebra $\mathbf{A}$ is a D-poset. We remark that each orthomodular poset (Example 1) is an orthoalgebra, but the opposite is not true [see example of R. Wright in the paper of Foulis *et al.* (1992)].

Example 4 (Kôpka and Chovanec, 1994; Dvurečenskij and Riečan, 1994). Let $\mathscr{E}(H)$ be the set of all Hermitian operators T on a Hilbert space H with $O \leq T \leq I$, where O and I are the zero and identity operators, respectively, on H. The set $\mathscr{E}(H)$ is a D-poset, which is not an orthoalgebra.

Example 5 (Kôpka, 1992). Let Ω be a nonempty set and $\mathscr{F}$ the family of all fuzzy sets on Ω, i.e., $\mathscr{F} = [0, 1]^{\Omega}$. We have for $f, g \in \mathscr{F}$

$$f \leq g \Leftrightarrow f(\omega) \leq g(\omega) \qquad (\omega \in \Omega)$$

Let Φ: $[0, 1] \to [0, \infty)$ be an injective increasing continuous function such that $\Phi(0) = 0$. Taking for $f \leq g$

$$(g \ominus f)(\omega) = \Phi^{-1}(\Phi(g(\omega)) - \Phi(f(\omega)) \qquad (\omega \in \Omega)$$

we obtain that $\mathscr{F}$ is a D-poset.

3. $\oplus$-ORTHOGONALITY

$\mathbf{L}$ will always denote a D-poset. Let $\{a_1, \ldots, a_n\} \subset \mathbf{L}$. We define

$$a_1 \oplus \cdots \oplus a_n = \mathbf{0} \qquad \text{for} \quad n = 0$$

$$a_1 \oplus \cdots \oplus a_n = a_1 \qquad \text{for} \quad n = 1$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \qquad \text{for} \quad n \geq 3$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $a_1 \oplus \cdots \oplus a_n$ exist in $\mathbf{L}$. We have, by Dvurečenskij and Riečan (1994), the following definition.

Definition 2. A finite subset $a_1 \oplus \cdots \oplus a_n$ of $\mathbf{L}$ is $\oplus$-orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists in $\mathbf{L}$.

We say that a $\oplus$-orthogonal subset $\{a_1, \ldots, a_n\}$ of $\mathbf{L}$ has a $\oplus$-sum, $\oplus_{i=1}^n a_i$, defined by

$$\bigoplus_{i=1}^{n} a_i := a_1 \oplus \cdots \oplus a_n$$

We remark that the preceding $\oplus$-sum is independent of any permutation of elements.

Definition 3. A subset G of $\mathbf{L}$ is $\oplus$-orthogonal if every finite subset F of G is $\oplus$-orthogonal.

We say that a $\oplus$-orthogonal subset $G = \{a_i: i \in I\}$ of $\mathbf{L}$ has a $\oplus$-sum in $\mathbf{L}$, $\oplus_{i \in I} a_i$, if in $\mathbf{L}$ there exists the join

$$\bigoplus_{i \in I} a_i := \sup\left\{\bigoplus_{i \in F} a_i \text{: } F \text{ finite subset of } I\right\}$$

Any subset of a $\oplus$-orthogonal set is again $\oplus$-orthogonal.

Definition 4. A D-poset $\mathbf{L}$ is a complete D-poset ($\sigma(\oplus)$–D-poset) if, for every $\oplus$-orthogonal subset (every countable $\oplus$-orthogonal subset) G of $\mathbf{L}$, there exists the $\oplus$-sum in $\mathbf{L}$.

Definition 5. A D-poset $\mathbf{L}$ is quasi-σ-complete if for every $\oplus$-orthogonal sequence $\{a_i\}$ in $\mathbf{L}$ there exists a subsequence $\{a_i\}_{i \in M}$ such that $\oplus_{i \in I} a_i \in \mathbf{L}$ for each $I \subset M$.

Remark 1. The notion of quasi-σ-ring was introduced by Constantinescu (1981, 1984).

We shall give now an example of a $\sigma(\oplus)$–D-poset.

Example 6 (de Lucia and Pap, 1995). Let S be any set of real numbers between 0 and 1 where S satisfies the following conditions:

(i) $0 \in S$ and $1 \in S$.
(ii) If $x, y \in S$, then $\min(1, x + y) \in S$.
(iii) If $x, y \in S$, then $\max(0, x + y - 1) \in S$.
(iv) If $x \in S$, then $1 - x \in S$.

The operations $\oplus$, $\odot$, and $*$ are defined as follows:

$$x \oplus y := \min(1, x + y)$$

$$x \odot y := \max(0, x + y - 1)$$

$$x^* := 1 - x$$

The system $(S, \oplus, \odot, *, 0, 1)$ is an MV-algebra. If we take $S = [0, 1]$, we obtain a σ-MV-algebra with respect to the operation $\oplus$ and in this way also a $\sigma(\oplus)$–D-poset, since for $x \leq y$ we have that the operation $\ominus$ defined by

$$x \ominus y := (x \oplus y^*)^*$$

gives a σ–D-poset with respect to the operation $\oplus_D$ defined by

$$x \oplus_D y = (y^* \ominus x)^*$$

which coincides with the operation $\oplus$, i.e.,

$$x \oplus_D y = (y^* \ominus x)^* = ((x \oplus (y^*)^*)^*)^* = x \oplus y$$

We remark that for $S = \{0, 1\}$ we trivially obtain also an $\sigma(\oplus)$–D-poset. But if $S =$ the set of all rational numbers between 0 and 1, then this is a MV-algebra, and so also a D-poset, which is not a $\sigma(\oplus)$–MV-algebra and so also not a $\sigma(\oplus)$–D-poset.

4. NIKODYM BOUNDEDNESS THEOREM

Let Y be an uniform space with the uniformity $\mathcal{U}$. We have, by Hejcman (1959), the following statement.

Definition 6. A subset B of Y is bounded ($\mathcal{U}$-bounded) if for every $U \in \mathcal{U}$ there exist a finite set $K \subset B$ and a natural number n such that

$$B \subset U^n[K]$$

where $U^1 = U$, $U^n = U \circ U^{n-1}$ ($\circ$ is the composition of the relations), and $U[K]$ is the set of all $x \in Y$ such that $(x, y) \in U$ for some $y \in K$.

A subset B of a metrizable uniform space $(Y, \mathcal{U})$ is $\mathcal{U}$-bounded if and only if it is d-bounded for every metric d generating the same uniformity $\mathcal{U}$. The following known characterization of $\mathcal{U}$-boundedness will be often used.

Theorem 1. A set $B \subset Y$ is $\mathcal{U}$-bounded if and only if it is d-bounded for every uniformly continuous pseudometric d defined on Y.

We denote by $\mathcal{D}$ the family of all uniformly continuous pseudometrics defined on $(Y, \mathcal{U})$.

Let $\mathbf{L}$ be a quasi-σ–D-poset.

Definition 7. For $d \in \mathcal{D}$ the d-semivariation of a function μ: $\mathbf{L} \to Y$ with respect to a point $x_0 \in Y$ is

$$\tilde{\mu}_d^{x_0}(b) := \sup\{d(\mu(c), x_0): c \leq b, c \in \mathbf{L}\} \qquad (b \in \mathbf{L})$$

We define for $d \in \mathcal{D}$ and $x_0 \in Y$ a function $\mu: \mathbf{L} \to Y$

$$\alpha_d^{x_0}(a, \mu) := \lim_{n\to\infty} \sup\left\{d(\mu(a \oplus b), x_0): \tilde{\mu}_d^{x_0}(b) < \frac{1}{n}, b \in \mathbf{L}\right\} \qquad (a \in \mathbf{L})$$

Theorem 2. Let $\mathcal{M}$ be a family of x_0-exhaustive functions $\mu: \mathbf{L} \to Y$, i.e., $\lim_{n\to\infty} d(m(a_n), x) = 0$ for each $\oplus$-orthogonal sequence $\{a_n\}$, where $\mathbf{L}$ is a quasi-σ–D-poset and Y is a uniform space. Then the set

$$\{\mu(a): \mu \in \mathcal{M}, a \in \mathbf{L}\}$$

is $\mathcal{U}$-bounded if and only if the following conditions hold:

(i) For each $d \in \mathcal{D}$ and each $m \in N$ there exists $s(m) \in N$ such that

$$d(\mu(a), \mu(b)) > s(m)$$

implies either, for $b \leq a$,

$$d(\mu(a \ominus b), x_0) > m$$

or, for $a \leq b$,

$$d(\mu(b \ominus a), x_0) > m$$

(ii) For each $d \in \mathcal{D}$ the set

$$\{\alpha_d^{x_0}(a, \mu): \mu \in \mathcal{M}\}$$

is bounded for each $a \in \mathbf{L}$.

(iii) For each $d \in \mathcal{D}$

$$\{d(\mu(a_n), x_0): \mu \in \mathcal{M}, n \in N\}$$

is bounded for every orthogonal sequence $\{a_n\}$ from $\mathbf{L}$.

Proof. The necessary part of the proof is obvious. Suppose that (i)–(iii) hold, but the set

$$\{\mu(a): \mu \in \mathcal{M}, a \in \mathbf{L}\}$$

is not $\mathcal{U}$-bounded. Then by Theorem 1 and (iii) there exist d from $\mathcal{D}$ and an orthogonal sequence $\{a_n\}$ from $\mathbf{L}$ such that

$$d(\mu_n(a_n), x_0) > n \qquad (n \in N)$$

We take by (ii)

$$m_1 := \sup_n \alpha_d^{x_0}(a_1, \mu_n) + 1 \in N \tag{1}$$

Then, there exists $s(m_1) + 1 \in N$, by (i). We take $n_1 > s(m_1) + 1$. By Lemma

1 from de Lucia and Pap (1995) and (1) there exists a subsequence $\{a_n^1\}$ of the sequence $a_{n_1+1}, a_{n_2+2}, \ldots$ such that

$$d\left(\mu_{n_1}\left(a_1 \oplus \bigoplus_{i\in I} a_i^1\right), x_0\right) < m_1$$

for arbitrary $I \subset N$. Now we take

$$m_2 := \sup_n \alpha_d^{x_0}(a_1 \oplus a_{n_1}, \mu_n) + 1 \in N$$

by (ii) and we repeat the whole preceding procedure.

Continuing this procedure, we obtain two sequences of natural numbers $\{m_k\}$ and $\{n_k\}$, $n_0 = 1$, such that for

$$m_k = \sup_n \alpha_d^{x_0}\left(\bigoplus_{i=0}^{k-1} a_{n_i}, \mu_n\right) + 1$$

we have

$$d(\mu_{n_k}(a_{n_k}), x_0) > n_k > s(m_k) + k \qquad (k \in N) \tag{2}$$

and

$$d\left(\mu_{n_k}\left(\bigoplus_{i=0}^{\infty} a_{n_i} \ominus a_{n_k}\right), x_0\right) < m_k \qquad (k \in N) \tag{3}$$

[by (i)].

Taking $k > \sup_j d(\mu_{n_j}(\oplus_{i=0}^{\infty} a_{n_i}), x_0)$, we obtain, by (2),

$$d\left(\mu_{n_k}\left(\bigoplus_{i=0}^{\infty} a_{n_i}\right), \mu_{n_k}(a_{n_k})\right)$$

$$\geq d(\mu_{n_k}(a_{n_k}), x_0) - d\left(\mu_{n_k}\left(\bigoplus_{i=0}^{\infty} a_{n_i}\right), x_0\right) > s(m_k)$$

Hence by condition (i)

$$d\left(\mu_{n_k}\left(\bigoplus_{i=0}^{\infty} a_{n_i} \ominus a_{n_k}\right), x_0\right) > m_k \qquad (k \in N)$$

Contradiction with (3).

If **L** is an orthomodular lattice (Example 1), then we can relax the conditions in the previous theorem.

Theorem 3. Let $\mathbf{L}$ be a σ-orthomodular lattice. Let $\mathcal{M}$ be a family of functions μ: $\mathbf{L} \to Y$. Then the set

$$\{\mu(a): \mu \in \mathcal{M}, a \in \mathbf{L}\}$$

is $\mathcal{U}$-bounded if and only if the following conditions hold:

(i) For each $\mu \in \mathcal{M}$ and each $m \in N$ there exists $s(m) \in N$ such that for each a, b, $\in \mathbf{L}$

$$d(\mu(a), \mu(b)) > s(m) \qquad (d \in \mathcal{D}, \mu \in \mathcal{M})$$

implies either

$$\tilde{\mu}_d^{x_0}((b \vee a')') > m$$

or

$$\tilde{\mu}_d^{x_0}((a \vee b')') > m$$

(ii) For each $d \in \mathcal{D}$ the set

$$\{\mu(a_n): \mu \in \mathcal{M}, n \in N\}$$

is d-bounded for every orthogonal sequence $\{a_n\}$ from $\mathbf{L}$.

Proof. The necessity of (i) and (ii) is obvious. Now, suppose that (i) and (ii) hold, but the set

$$\{\mu(a): a \in \mathbf{L}, \mu \in \mathcal{M}\}$$

is not $\mathcal{U}$-bounded. Then for any $x_0 \in Y$ there exist $d \in \mathcal{D}$, a sequence $\{\mu_n\}$ from $\mathcal{M}$, and a sequence $\{a_n\}$ of elements from $\mathbf{L}$ such that

$$d(\mu_n(a_n), x_0) > n \qquad (n \in N) \tag{4}$$

On the other hand, condition (ii) implies that there exists $M > 0$ such that

$$d(\mu_n(a_1), x_0) < M$$

Hence by (4)

$$d(\mu_n(a_1), \mu_n(a_n)) \to \infty$$

as $n \to \infty$. Then, for $n \to \infty$, by (i) either

(a) $\sup\{d(\mu_n(c), x_0): c \in \mathbf{L}, c \leq (a_1 \vee a_n')'\} \to \infty$

or

(b) $\sup\{d(\mu_n(c), x_0): c \in \mathbf{L}, c \leq (a_n \vee a_1')'\} \to \infty$

Case (a). We shall choose a subsequence $\{\mu_{n_k}\}$ of the sequence $\{\mu_n\}$ and a sequence $\{e_n\}$ of orthogonal elements from $\mathbf{L}$ such that

$$d(\mu_{n_k}(e_k), x_0) > k \qquad (k \in N)$$

Let $\mu_{n_1} = \mu_1$ and $e_1 = a_1$. We choose n_2 such that $\mu_{n_2} = \mu_{i_2}$, where i_2, i_3, . . . is a sequence of natural numbers, elements $a_{i_2}, a_{i_3}, \ldots$ from the sequence $\{a_n\}$, and elements a_j^1 from **L** such that $a_j^1 \le a_{i_j}$ and

$$d(\mu_{i_j}((a_1 \vee (a_j^1)')'), x_0) > j, \qquad j = 2, 3, \ldots$$

[contradiction with (ii)]. We take $e_2 = (a_1 \vee (a_2^1)')'$. Now we repeat the preceding procedure, taking the sequences $\{\mu_{i_j}\}$ and $\{(a_1 \vee (a_j^1)')'\}_2^\infty$. Continuing this procedure, we obtain the sequences $\{\mu_n\}$ and $\{e_n\}$, which give a contradiction with (ii).

Case (b). Let $\mu_{n_1} = \mu_1$. We choose n_2 such that $\mu_{n_2} = \mu_{i_2}$, where we take a sequence $i_2, i_3, \ldots$ of natural numbers, a sequence of elements $a_{i_2}, a_{i_3}, \ldots$ from the sequence $\{a_n\}$, and elements a_j^1 ($j = 2, 3, \ldots$) from **L** such that $a_j^1 \ge a_{i_j}$ and

$$d(\mu_{i_j}((a_j^1 \vee a_1')'), x_0) > j, \qquad j = 2, 3, \ldots$$

Let $b_1 = a_1$ and $b_2 = (a_2^1 \vee a_1')'$. Now we shall repeat the procedure for the sequences $\mu_{i_2}, \mu_{i_3}, \ldots$ and $(a_2^1 \vee a_1')', (a_3^1 \vee a_1')', \ldots$. The sequence $\{b_n\}$ is decreasing, i.e., $b_n \ge b_{n+1}$ ($n \in N$). The condition (i) implies the existence of the subsequences $\{\mu_{n_{k_i}}\}$ and $\{b_{k_i}\}$ such that

$$(\tilde{\mu}_{n_{k_i}})_d^{x_0}((b_{k_i} \vee b'_{k_{i-1}})') > i \qquad (i \in N)$$

Since we have

$$(b_{k_i} \vee b'_{k_{i-1}})' = b'_k \wedge b_{k_{i-1}}$$

the sequence $\{(b_{k_i} \wedge b'_{k_{i-1}})'\}$ is orthogonal. Hence the preceding inequalities imply the existence of an orthogonal sequence $\{c_i\}$ such that $c_i \le (b_{k_i} \vee b'_{k_{i-1}})'$ and

$$d(\mu_{n_{k_i}}(c_i), x_0) > i \qquad (i \in N)$$

Contradiction with (ii).

REFERENCES

Antosik, P., and Swartz, C. (1985). *Matrix Methods in Analysis,* Springer, Berlin.

Birkhoff, G. (1967). *Lattice Theory,* 3rd ed., American Mathematical Society, Providence, Rhode Island.

Chang, C. C. (1958). *Transactions of the American Mathematical Society,* 467–490.

Constantinescu, C. (1981). *Libertas Mathematica*, **1**, 51–73.
Constantinescu, C. (1984). *Spaces of Measures*, de Gruyter, Berlin.
Cook, A. T. (1978). The Nikodym–Hahn–Vitale–Saks Theorem for states on a quantum logic, in *Proceedings of Mathematical Foundations of Quantum Theory*, Academic Press, New York.
De Lucia, P., and Dvurečenskij, A. (1993a). *Tatra Mountains Mathematical Publications*, **2**, 229–239.
De Lucia, P., and Dvurečenskij, A. (1993b). *Tatra Mountains Mathematical Publications*, **3**, 101–110.
De Lucia, P., and Morales, P. (1992). Decomposition theorems in Riesz spaces, Preprint University of Naples.
De Lucia, P., and Pap, E. (1995). Nikodym convergence theorem for uniform space valued functions defined on *D*-poset, *Mathematica Slovaca*, **45**, to appear.
De Lucia, P., and Pap, E. (n.d.). Diagonal theorems and their applications, to appear.
Dunford, N., and Swartz, J. (1958). *Linear Operators I*, Interscience, New York.
Dvurečenskij, A. (1988). *Letters in Mathematical Physics*, **15**, 231–235.
Dvurečenskij, A. (1991). Regular measures and completeness of inner product spaces, in *Contributions to General Algebras*, Vol. 7, Hölder-Pichler-Tempski Verlag, pp. 137–147.
Dvurečenskij, A. (1993). *Gleason's Theorem and Applications*, Kluwer, Dordrecht, and Ister Science Press, Bratislava.
Dvurečenskij, A. (n.d.). Tensor product of difference posets, *Transactions of the American Mathematical Society*, to appear.
Dvurečenskij, A., and Pulmannova, S. (1994a). *International Journal of Theoretical Physics*, **33**, 819–850.
Dvurečenskij, A., and Pulmannova, S. (1994b). *Reports on Mathematical Physics*, **34**, 151–170.
Dvurečenskij, A., and Riečan, B. (1991). *International Journal of General Systems*, **20**, 39–54.
Dvurečenskij, A., and Riečan, B. (1994). *International Journal of Theoretical Physics*, **33**, 1387–1402.
Foulis, D. J., Greechie, R. J., and Rüttimann, G. T. (1992). *International Journal of Theoretical Physics*, **31**, 787–807.
Guarigilia, E. (1990). *Acta Scientiarum Mathematica*, **54**, 391–407.
Hejcman, H. (1959). *Czechoslovak Mathematical Journal*, **9**, 544–563.
Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.
Kalmbach, G., and Riečanova, Z. (n.d.). An axiomatization for Abelian relative inverses, to appear.
Klement, E. P., and Weber, S. (1991). *Fuzzy Sets and Systems*, **40**, 375–394.
Klimkin, V. M. (1989). *Matematičeskij Sbornik*, **180**(3), 385–396.
Kôpka, F. (1992). *Tatra Mountains Mathematical Publications*, **1**, 83–87.
Kôpka, F., and Chovanec, F. (1994). *Mathematica Slovaca*, **44**, 21–34.
Luxemburg, W. A. J., and Zaanen, A. C. (1971). *Riesz spaces I*, North-Holland, Amsterdam.
Mesiar, R. (1994). Fuzzy difference posets and MV-algebras, in *Proceedings IPMU'94*, B. Bouchon-Meuniere and R. R. Yager, eds., Paris, pp. 202–206.
Mesiar, R. (n.d.). Differences on [0, 1], *Tatra Mountains Mathematical Publications*, **6**, to appear.
Mundici, D. (1986). *Journal of Functional Analysis*, **65**, 15–53.
Navara, M. (n.d.). An orthomodular lattice admitting no group-valued measure, *Proceedings of the American Mathematical Society*, to appear.
Navara, M., and Ptak, P. (n.d.). Difference posets and orthoalgebras, to appear.
Pap, E. (1982). *Functional Analysis*, Institute of Mathematics, Novi Sad, Yugoslavia.
Pap, E. (1986). *Acta Scientiarum Mathematica*, **50**, 159–167.

Pap, E. (1988). *Univerzitet u Novom Sadu Zbornik Radova Prirodno-Matematičkog Fakulteta Serija Matematika,* **18**, 101–109.

Pap, E. (1991a). *Atti Seminario Matematica Fisica Universita Modena,* **39**, 345–360.

Pap, E. (1991b). *Univerzitet u Novom Sadu Zbornik Radova Prirodno-Matematičkog Fakulteta Serija Matematika,* **21**(1), 75–82.

Pap, E. (1994). *Fuzzy Sets and Systems,* **65**, 71–83.

Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics,* Kluwer, Dordrecht.

Randall, C., and Foulis, D. (1979). New definitions and theorems, University of Massachusetts Mimeographed Notes, Amherst, Massachusetts.

Randall, C., and Foulis, D. (1981). Empirical logic and tensor products, in *Interpretations and Foundations of Quantum Theory,* Vol. 5, H. Neumann, ed., Wissenschaftsverlag, Bibliographisches Institut, Mannheim, pp. 9–20.

Riečanova, Z., and Bršel, D. (1994). *International Journal of Theoretical Physics,* **33**, 133–141.

Swartz, C. (1992). *Introduction to Functional Analysis,* Dekker, New York.